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POINCARÉ-MELNIKOV THEORY OF HOMOCLINICS AND CHAOS: A VARIATIONAL APPROACH *

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1. THE ABSTRACT THEOREM.

Let E be a Hilbert space and let $f_0 : E \to \mathbb{R}$, $G : \mathbb{R} \times E \to \mathbb{R}$ satisfy

1. $f_0 \in C^2(E, \mathbb{R})$;
2. f_0 has a d-dimensional manifold of critical points Z. For the sake of simplicity, we will suppose that $Z = \{z(\theta) : \theta \in U\}$, U open subset of $\mathbb{R}^d$;
3. $\forall\, z \in Z$, $f_0''(z)$ is Fredholm index 0;
4. $\forall\, z \in Z$, $Ker[f_0''(z)] = T_z Z$ ($T_z Z$ denotes the tangent space to Z at z);
5. $G(0, u) = 0$ for all $u \in E$;
6. G is C^2 with respect to u;
7. the maps $(\varepsilon, u) \mapsto G(\varepsilon, u)$, $(\varepsilon, u) \mapsto D_u G(\varepsilon, u)$, $(\varepsilon, u) \mapsto D^2_{uu} G(\varepsilon, u)$ are continuous.
8. there exist $\alpha > 0$ and $\Gamma \in C(U, \mathbb{R})$ such that
$$\varepsilon^{-\alpha} G(\varepsilon, z(\theta)) \to \Gamma(\theta), \quad D_u G(\varepsilon, z(\theta)) = o(\varepsilon^{\alpha/2}), \quad \text{as } \varepsilon \to 0.$$

Let $R > 0$ and $\theta_0 \in U$ be such that $\Gamma(\theta_0) < \inf\{\Gamma(\theta) : |\theta| = R\}$.

Then there exists $\varepsilon_0 > 0$ such that for all $|\varepsilon| < \varepsilon_0$ the perturbed functional
$$f_\varepsilon(u) = f_0(u) + G(\varepsilon, u)$$
has a critical point $u_\varepsilon = z(\theta_\varepsilon) + O(\varepsilon)$, with $|\theta_\varepsilon| < R$.

Furthermore, if Γ has a (possibly degenerate) isolated minimum (maximum) at some $\bar\theta \in U$ then $\theta_\varepsilon \to \bar\theta$ and hence f_ε has a critical point u_ε such that $u_\varepsilon \to z(\bar\theta)$.

For details and other results we refer to [2, 3], see also [6].

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2. APPLICATIONS.

We list some specific examples of applications of the preceding abstract result. The general cases are discussed in the papers cited below.

- **Homoclinics of dynamical systems** ([2]). Consider an equation like

$$q'' - q + V'(q) = \varepsilon W_q'(t,q) \tag{A}$$

where, roughly, $V(q) \simeq |q|^a$, $a > 2$ and $W(t,0) = W_q'(t,0) = 0$ (the case that $W(t,q) = h(t)q$ can also be handled). Set: $E = H^1(\mathbb{R})$, $||u||^2 = \int_{\mathbb{R}}(|u'|^2 + |u|^2)dt$,

$$f_0(u) = \frac{1}{2}||u||^2 - \int_{\mathbb{R}} V(u)dt,$$

and

$$G(\varepsilon,u) = \varepsilon \int_{\mathbb{R}} W(t,u)dt.$$

Functionals f_0 and G satisfy $1-8$ with $d=1$ and $\alpha = 1$. If $\phi(t)$ is a solution of the unperturbed equation

$$q'' - q + V'(q) = 0$$

then one has

$$\Gamma(\theta) = \int_{\mathbb{R}} W(t, \phi(t+\theta))dt$$

which is the classical Poincaré function, or else the primitive of the Melnikov function.

Similar results hold for the PDE analogous of (A), a case in which the critical manifold Z has dimension $d > 1$.

- **Multibump Solutions** ([8]). If $\Gamma(\theta)$ oscillates it is possible to "glue together" two or more homoclinics to find multibump solutions of (A). More precisely, using the fact that $q = 0$ is an hyperbolic equilibrium one can show the existence of solutions with infinitely many bumps, located near any *prescribed* minima (or maxima) of Γ. In particular, this implies that the dynamical system has positive topological entropy and a complicated behaviour. The oscillation of Γ arises, for example, when W is (periodic or) almost periodic in t.

In the classical approach the preceding results are usually obtained under the assumption that the Melnikov function Γ' has a simple zero.

- **Heteroclinics** ([7]). Consider an equation like (A) and suppose that V is a double-well potential. If the unperturbed problem has a heteroclinic, then one can still use the abstract approach to find heteroclinics of (A) provided the Poincaré function Γ has a minimum or maximum. Furthermore, using the fact that the system is reversible, one can find multibump solutions and a complex dynamics.

- **Slowly oscillating systems** ([5]). Consider an equation like

$$q'' - q + |q|^{p-1}q = h(\varepsilon t)W'(q) \tag{B}$$

where $W(0) = W'(0) = W''(0) = 0$. If h is bounded and has a local minimum (or maximum) at some $t = \tau_0$ then (B) has a homoclinic solution $u_\varepsilon(t) \simeq \phi(t - \tau_0/\varepsilon)$, where ϕ denotes a homoclinic of the unperturbed equation

$$q'' - q + |q|^{p-1}q = 0.$$

If h has infinitely many minima (or maxima) at τ_i with

$$0 < const. < \inf(\tau_{i+1} - \tau_i) < \sup(\tau_{i+1} - \tau_i) < Const. < \infty,$$

then there exist multibump solutions yielding a complex dynamics. One can also handle the case that h is flat near the minima (maxima).

- **Bifurcation of bound states from the essential spectrum** ([3]). Consider an equation like

$$\left\{ \begin{array}{rcl} \psi'' + \lambda\psi + h(x)|\psi|^{p-1}\psi & = & 0, \\ \lim_{|x|\to\infty} \psi(x) & = & 0. \end{array} \right. \tag{C}$$

Setting (for $\varepsilon \neq 0$) $u(x) = \varepsilon^{2/(1-p)}\psi(x/\varepsilon)$ and $\lambda = -\varepsilon^{-2}$, (C) becomes

$$u'' - u + h(x/\varepsilon)|u|^{p-1}u = 0. \tag{C'}$$

If $h(x) \to L$ as $|x| \to \infty$ equation (C') can be considered as a perturbation of

$$u'' - u + L \cdot |u|^{p-1}u = 0.$$

Here one has that

$$G(\varepsilon, u) = \frac{1}{p+1} \int_{\mathbb{R}} [L - h(x/\varepsilon)]\, |u|^{p+1} dx.$$

The family u_ε of solutions of (C') give rise to solutions $\psi_\varepsilon(x) = \varepsilon^{2/(p-1)}u_\varepsilon(\varepsilon x)$ of (D) that converge to zero as $\varepsilon \to 0$. In addition, since $\lambda = -\varepsilon^2 \to 0$, they branch off from the infimum of the essential spectrum.

- **Semilinear Schrödinger equations** ([4]). Consider

$$\left\{ \begin{array}{rcl} -\varepsilon^2 u'' + u + Q(x)u & = & |u|^{p-1}u, \\ \lim_{|x|\to\infty} u(x) & = & 0 \end{array} \right.$$

where $x \in \mathbb{R}$, $p > 1$ and Q is a bounded potential with a proper local minimum (or maximum) at $x = 0$, with $Q(0) = 0$ (if $x \in \mathbb{R}^n$, $n > 2$, one requires that $p < (n+2)/(n-2)$). After rescaling one finds

$$u'' - u + |u|^{p-1}u = Q(\varepsilon x)u.$$

Here $G(\varepsilon, u) = \int Q(\varepsilon x)u^2 dx$ and assumtion 7 does not hold. However, a suitable modification of the abstract setting yields solutions u_ε such that $u_\varepsilon(x) \simeq \phi(x/\varepsilon)$, as well as multi-bump solutions.

- In several cases it is possible to evaluate the (generalized) Morse index of the critical points of f_ε. In applications this permits to study the orbital stability of the solutions. See, for example, [1] and [4].

Other applications deal with the existence of asymmetric bound states for equations arising in nonlinear optics, see [1] and with the existence of solutions of problems at resonance on $\mathbb{R}^n$, see [9, 10].

References

[1] A. Ambrosetti - D. Arcoya - J. Gámez. Asymmetric bound states of differential equations in nonlinear optics. To appear

[2] A. Ambrosetti - M. Badiale, Homoclinics: Poincaré-Melnikov type results via a variational approach. *C. R. Acad. Sci. Paris*, t. 323, Série I (1996), 753-758, and *Annales I.H.P. - Analyse nonlin.*, to appear.

[3] A. Ambrosetti - M. Badiale, Variational perturbative methods and bifurcation of bound states from the essential spectrum. Preprint S.N.S. 1997.

[4] A. Ambrosetti - M. Badiale - S. Cingolani, Semiclassical states of nonlinear Schroedinger equations. *Rend. Mat. Acc. Lincei*, s.9 v.7 (1996), 155-160, and *Archive Rat. Mech. Anal.*, 1997.

[5] A. Ambrosetti - M. Berti, Homoclinics and complex dynamics in slowly oscillating systems. To appear on *Cont. Discr. Dyn. Systems.*

[6] A. Ambrosetti - V. Coti Zelati - I. Ekeland, Symmetry breaking in Hamiltonian systems. *Jour. Diff. Equat.* **67** (1987), 165-184.

[7] M. Berti, Heteroclinic solutions for perturbed second order systems. *Rend. Mat. Acc. Lincei*, to appear.

[8] M. Berti - P. Bolle, Homoclinics and chaotic behaviour for perturbed second order systems. Preprint S.N.S, Pisa, 1997.

[9] M. Badiale - B. Pellacci - S. Villegas, Elliptic problems on $\mathbb{R}^n$ with jumping nonlinearities: perturbation results. In preparation.

[10] B. Pellacci - S. Villegas, Some existence results for a class of resonant problems on $\mathbb{R}^n$. Preprint S.N.S. 1997.